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Solution of Coupled Differential Equations of Soliton Theory

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We present the complete set of solutions of the coupled differential equations of the form $(\nabla\alpha)^2=\gamma(\alpha)$, $\nabla^2\alpha=\delta(\alpha)$. Equations of this form appear in several physical situations.

1. INTRODUCTION

We study the equations

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=\gamma(\alpha) \tag{1.1a}$$

$$\alpha_{11}+\alpha_{22}+\alpha_{33}=\delta(\alpha) \tag{1.1b}$$

where $\alpha_1=\partial\alpha/\partial x^1$, $\alpha_{11}=\partial^2\alpha/\partial x^2$, etc., and $\gamma(\alpha)$ and $\delta(\alpha)$ are functions of α.

Equations of this type occur in two cases, one in the special case of the nonlinear field equations for the chiral invariant model of pion dynamics studied by Ray (1978), and another in the special situation of the problem of the stability of a scalar soliton studied by Schiff (1982). Further, the form of the equations indicates that they may occur in other physical situations, too. Also, it is interesting to note that if α satisfies a set of coupled equations of the form (1.1), then any function of α also satisfies a coupled equation of the form (1.1). For these reasons an attempt to get solutions of the equations (1.1) seems worthwhile. Some particular solutions were given by Ray and Schiff in the two cases.

The coupled equations (1.1a) and (1.1b) can be equivalently written as

$$\Theta_1^2+\Theta_2^2+\Theta_3^2=1 \tag{1.1a$'$}$$

$$\Theta_{11}+\Theta_{22}+\Theta_{33}=\sigma(\Theta) \tag{1.1b$'$}$$

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where

$$\Theta = \int \gamma^{-1/2}\, d\alpha \tag{1.2a}$$

and $\sigma(\Theta)$ is a function of Θ, related to σ and γ by the relation

$$\sigma = \gamma^{-1/2}(\delta - \tfrac{1}{2}) \tag{1.2b}$$

In this paper we find the complete set of solutions of the coupled equations (1.1a′) and (1.1b′).

2. SOLUTIONS

Equation (1.1a′) admits three classes of solutions, which as given in Forsyth (1906) are:

(i) $$\Theta = Ax^1 + Bx^2 + Cx^3 + D \tag{2.1a}$$

where D is an arbitrary constant and A, B, C are constants given by

$$A^2 + B^2 + C^2 = 1 \tag{2.1b}$$

(ii) $$\Theta = ax^1 + bx^2 + cx^3 + f(a, b) \tag{2.2a}$$

where $f(a, b)$ is an arbitrary function and a, b, c are functions of x^1, x^2, and x^3 given by

$$-x^1 + \frac{a}{c}x^3 = \frac{\partial f}{\partial a} \tag{2.2b}$$

$$-x^2 + \frac{b}{c}x^3 = \frac{\partial f}{\partial b} \tag{2.2c}$$

$$a^2 + b^2 + c^2 = 1 \tag{2.2d}$$

(iii) $$\Theta = px^1 + qx^2 + mx^3 + g(q) \tag{2.3a}$$

where $g(q)$ is an arbitrary function and p, q, and m are functions of x^1, x^2, and x^3 given by

$$p = h(q) \tag{2.3b}$$

$$\frac{dg}{dq} + \left(x^1 - \frac{p}{q}x^3\right)\frac{dh}{dq} + x^2 - \frac{q}{m}x^3 = 0 \tag{2.3c}$$

$$p^2 + q^2 + m^2 = 1 \tag{2.3d}$$

So our task is to choose the arbitrary constant D and arbitrary functions f and g (and so automatically evaluate each of A, B, C, a, b, c, p, q, m) in such a way that equations (2.1), (2.2), and (2.3) separately satisfy (1.1b′).

Case i. Solution from (2.1). In this case, putting (2.1) into (1.1b′), we see that the solution is possible only if

$$\sigma = 0 \tag{2.4}$$

In that case the solution for the coupled equations (1.1a′) and (1.1b′) exists only if $\sigma = 0$ and that solution is given by (2.1) itself.

Case ii. Solution from (2.2). Here, putting (2.2) in (1.1b′), one gets after little calculation

$$\frac{(1-X^2-Y^2)[f_{XX}f_{YY}-Z^2/(1-X^2-Y^2)]}{2Z-(1-X^2)f_{XX}-(1-Y^2)f_{YY}+2XYf_{XY}}+(Xf_X+Yf_Y)-f=F(\Theta) \tag{2.5a}$$

and

$$\Theta = Z - Xf_X - Yf_Y + f \tag{2.5b}$$

where

$$X = a, \qquad Y = b, \qquad Z = x^3/C \tag{2.5c}$$

and

$$F(\Theta) = 1/\sigma(\Theta) - \Theta \tag{2.5d}$$

From (2.5a) and (2.5b) we get

$$\begin{aligned}(1-X^2-Y^2)&(f_{XX}f_{YY}-f_{XY})-Z^2\\ &+(Xf_X+Yf_Y-f)[2Z(1-X^2)f_{XX}-(1-Y^2)f_{YY}+2XYf_{XY}]\\ &=[2Z-(1-X^2)f_{XX}-(1-Y^2)f_{YY}+2XYf_{XY}]F(Z-Xf_X-Yf_Y+f)\end{aligned} \tag{2.6}$$

where X, Y, Z are independent variables and f is a function of X and Y only.

Since the right-hand side of (2.6) is quadratic in Z and the first term of the left-hand side of (2.6) is linear in Z, it follows that

$$F(Z-Xf_X-Yf_Y+f) = -\tfrac{1}{2}(Z-Xf_X-Yf_Y+L) \tag{2.7}$$

where L is a constant.

Putting (2.7) back in (2.6) and equating the coefficients of Z and the terms free from Z separately to zero, one gets a pair of equations,

$$(1-X^2)f_{XX}+(1-Y^2)f_{YY}-2XYf_{XY}=2(Xf_X+Yf_Y-f-2L) \tag{2.8a}$$

$$(1-X^2-Y^2)(f_{XX}f_{YY}-f_{XY}^2)=(Xf_X+Yf_Y-f-2L) \tag{2.8b}$$

Equations (2.8a) and (2.8b) can be transformed to

$$(1-r^2)f_{rr}+\frac{1}{r^2}(f_{\theta\theta}+rf_r)-2(rf_r-r-2L)=0 \tag{2.9a}$$

$$(1-r^2)[-(rf_\theta-f_\theta)^2+r^2f_{rr}(f_{\theta\theta}+rf_r)]=r^4(rf_r-f-2L)^2 \tag{2.9b}$$

where

$$r^2 = X^2 + Y^2, \qquad \tan\theta = Y/X \tag{2.9c}$$

Combining equations (2.9a) and (2.9b), one gets

$$(1-r^2)(rf_{r\theta} - f_\theta)^2 + r^4[rf_r - f - 2L - (1-r^2)f_{rr}]^2 = 0 \tag{2.10}$$

Since in view of (2.2d) and (2.5c), $1 - r^2$ is positive, it follows from (2.10) that

$$rf_{r\theta} - f_\theta = 0 \tag{2.11a}$$

$$(1-r^2)f_{rr} - rf_r + f + 2L = 0 \tag{2.11b}$$

Solving (2.11a) and substituting it in (2.11b), we get

$$f = L_1 X + L_2 Y + L_3(1 - X^2 - Y^2)^{1/2} - 2L$$

as the solution of (2.8).

Hence $f(a, b)$ in (2.2a) is given by

$$f(a, b) = L_1 a + L_2 b + L_3(1 - a^2 - b^2)^{1/2} - 2L \tag{2.12}$$

where L_1, L_2, L_3 and L are constants.

Putting (2.12) into (2.2b) and (2.2c) and using (2.2d), we can find a, b, and c as functions of x^1, x^2, and x^3. Substituting these expressions for a, b, c, and (2.12) in (2.2a), we finally get the solution of the coupled equation (1.1a) and (1.1b′) given by

$$\Theta = ([(x^1 + L_1)^2 + (x^2 + L_2)^2 + (x^3 + L_3)^2]^{1/2} - 2L \tag{2.13}$$

where L_1, L_2, L_3 and L are constants and σ is given by (2.5d).

Case (iii). Solution from (2.3). Here, putting (2.3) in (1.1b′), one gets

$$\begin{aligned}
&\frac{d^2p}{dq^2}x^1 + \left[-\frac{d}{dq}\left(\frac{p}{(1-p^2-q^2)^{1/2}}\right)\frac{dp}{dq} - \frac{p}{(1-p^2-q^2)^{1/2}}\frac{d^2p}{dq^2} \right.\\
&\quad \left. - \frac{d}{dq}\frac{p}{(1-p^2-q^2)^{1/2}}\right] x^3 + \left.\frac{d^2g}{dq^2}\right\}\\
&\quad \times \left\{ -\left(\frac{dp}{dq}\right)^2 - 1 + \frac{d}{dq}[(1-p^2-q^2)^{1/2}]\left(q + p\frac{dp}{dq}\right)\frac{1}{(1-p^2-q^2)^{1/2}}\right\}^{-1}\\
&\quad = \frac{1}{\sigma(\Theta)}
\end{aligned} \tag{2.14a}$$

and

$$\Theta = \left(p - q\frac{dp}{dq}\right)x^1 + \left(\frac{pq}{(1-p^2-q^2)^{1/2}}\frac{dp}{dq} + \frac{1-p^2}{(1-p^2-q^2)^{1/2}}\right)x^3 + g - q\frac{dg}{dq} \tag{2.14b}$$

If (2.14a) and (2.14b) are to hold simultaneously, the lhs of (2.14a) has to be a function of the rhs of (2.14b). If, instead of taking x^1, x^2, and x^3 as three independent variables we regard x^1, x^3, and q as three independent variables (where p is a function of q), it is easy to see that the lhs of (2.14a) can only be a linear function of (2.14b). Therefore

$$\frac{1}{\sigma(\Theta)} = K_1\Theta + K_2 \tag{2.15}$$

where K_1 and K_2 are constants.

Since x^1, x^3, and q are independent, it follows from (2.14a), (2.14b), and (2.15) that

$$\begin{aligned}
\frac{d^2p/dq^2}{p - q\,dp/dq} &= \left[-\frac{d}{dq}\left(\frac{p}{(1-p^2-q^2)^{1/2}}\right)\frac{dp}{dq} - \frac{p}{(1-p^2-q^2)^{1/2}}\frac{d^2p}{dq} \right. \\
&\quad \left. - \frac{d}{dq}\left(\frac{p}{(1-p^2-q^2)^{1/2}}\right)\right] \\
&\quad \times \left[(1-p^2-q^2)^{1/2} + \frac{q^2}{(1-p^2-q^2)^{1/2}} + \frac{pq}{(1-p^2-q^2)^{1/2}}\frac{dp}{dq}\right]^{-1} \\
&= \frac{d^2g/dq^2 - K_2\Delta_1}{g - q\,dg/dq}
\end{aligned} \tag{2.16a}$$

where

$$\Delta_1 = -\left(\frac{dp}{dq}\right)^2 - 1 + \frac{d}{dq}[(1-p^2-q^2)^{1/2}]\left(q + p\frac{dp}{dq}\right)\frac{1}{(1-p^2-q^2)^{1/2}} \tag{2.16b}$$

and K_2 is a constant.

Equation (2.16a) is equivalently written as

$$\frac{p_{qq}}{p - qp_q} = \frac{(1-p^2-q^2)^{1/2}_{qq}}{(1-p^2-q^2)^{1/2} - q(1-p^2-q^2)^{1/2}_q} \tag{2.17a}$$

and

$$\frac{p_{qq}}{p - qp_q} = \frac{g_{qq} - K_2\Delta_1}{g - qg_q} \tag{2.17b}$$

where $p_q \equiv dp/dg$, $p_{qq} \equiv d^2p/dq^2$, etc.

Solving (2.17a) and (2.17b), we obtain

$$(1-p^2-q^2)^{1/2} = K_3 p + K_4 q \tag{2.18a}$$

and

$$g=\int\left(K_2\int\frac{\Delta_1}{qp_q-p}\,dq+K_5\right)\frac{qp_q-p}{q^2}\,dq+K_6 \tag{2.18b}$$

where K_2, K_3, K_4, K_5, and K_6 are constants and Δ_1 is given by (2.16b). Evaluating the integral on the right-hand side of (2.18b) with the use of (2.18a), one gets

$$g=K_5\frac{p}{q}-K_2(K_3^2+K_4^2+1)(K_3^2+1)\left\{\frac{p(K_3^2+1)+qK_3K_4}{q(K_3^2+K_4^2+1)^{1/2}}\right.$$
$$\left.\times\tan^{-1}\left(\frac{p(K_3^2+1)+qK_3K_4}{q(K_3^2+K_4^2+1)^{1/2}}\right)-\frac{1}{2}\log\left[1+\left(\frac{p(K_3^2+1)+qK_3K_4}{q(K_3^2+K_4^2+1)^{1/2}}\right)^2\right]\right\}+K_6 \tag{2.19}$$

Substituting the expression for g from (2.19) into the equation (2.30) and using (2.18a), we obtain

$$\pm\left[K_2^2(K_3^2+K_4^2+1)^{1/2}(K_3^2+1)\right.$$
$$\left.\times\tan^{-1}\left(\pm\frac{[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}}{q(K_3^2+K_4^2+1^{1/2}}\right)+K_5\right]$$
$$\times(K_3^2+1)-\{(K_3^2+K_4^2+1)q\pm K_3K_4[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}\}$$
$$\times q^2x^1\pm(K_3^2+1)q^2[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}x^2\mp q^2$$
$$\mp q^2\{-K_3q(K_3^2+K_4^2+1)\pm K_3[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}\}x^3=0 \tag{2.20}$$

Substituting the expression for g from (2.19) into (2.3a) and using (2.18a), one gets the complete solution of the equations (1.1a′) and (1.1b′) represented by

$$\Theta=q(qK_3K_4\{-1\pm(K_3^2+1)[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]\}$$
$$+[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}[\pm1+(K_3^2+1)(K_3^2+K_4^2+1)q^2])x^1$$
$$+(K_3^2+1)q^2\{1\mp(K_3^2+1)[(K_3^2+K_4^2+1)q^2]\}x^2$$
$$+(K_3^2+1)q^2(K_4\pm[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}\{K_3q(K_4^2+1)$$
$$\pm K_4[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}\})x^3+\{-K_3K_4q\mp(K_3^2+1)$$
$$\times[(K_3^2+1)-(K_3^2+K_4^2+1)q^2]^{1/2}\}-K_2(K_3^2+1)^2(K_3^2+K_4^2+1)$$
$$\times q\log q+K_6(K_3^2+1)q \tag{2.21}$$

where K_2, K_3, K_4, and K_6 are all constants and q is given by (2.20) and σ is given by (2.15).

3. CONCLUSION

In summary, we have obtained three types of solutions of the coupled equations (1.1′). The first is given by (2.1), in which Θ is a function of $Ax^1 + Bx^2 + Cx^3$, and the second type of solution is represented by (2.13), where Θ is a function of $(x^1 + L_1)^2 + (x^2 + L_2)^2 + (x^3 + L_3)^2$ and A, B, C, L_1, L_2, and L_3 are constants. These are the two types of solutions that could have been anticipated. But in (2.21) we find another type of solution which could not as easily have been anticipated, as the expression for Θ is formidable.

Solutions of equation (1.1) can be obtained from the solution of equations (1.1′) by using the relation (1.2a). We note that if α is any solution of (1.1′), then any function of α is a solution of (1.1′). These solutions can now be applied to the physical situations mentioned by Ray (1978) and Schiff (1982) and also possibly for other physical situations.

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